Borel Complexity of Archimedean Orders

Antoine Poulin

April 19, 2022

Left-Orderable groups

Throughout, Γ is an infinite group.

Definition

A total order on a group Γ is a **left-order** if

$$g < h \Rightarrow kg < kh.$$

We say that Γ is **left-orderable** if there is some left-order on it.

Examples

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are all left-orderable groups.
- If H, K are left-orderable groups and

$$1 \to K \to \Gamma \to H \to 1$$

  Γ is left-orderable.
- $\Gamma = \langle x, y \mid yxy^{-1} = x^{-1} \rangle$ is a semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}$, hence left-orderable.

Different Characterization

Definition

We say that $P \subset \Gamma$ is a **positive cone** if

- $P \cdot P \subset P$
- $P \sqcup P^{-1} = \Gamma - \{1\}$

Proposition

If $<$ is a left-order on Γ,

$$P_< := \{g \in \Gamma : g > \mathrm{id}\}$$

is a positive cone.

Proof

If $g, h > \mathrm{id}$,

$$\mathrm{id} < h \Rightarrow \mathrm{id} < g < gh$$

Hence $P_< \cdot P_< \subset P_<$.

Since $<$ is a total order, exactly one of $g = \mathrm{id}, g > \mathrm{id}$ or $g < \mathrm{id}$ is true. In the last case,

$$g < \mathrm{id} \Rightarrow \mathrm{id} < g^{-1},$$

Hence $P_< \sqcup P_<^{-1} = \Gamma - \{1\}$.

Duality between Orders and Cones

On the other hand, if P is a positive cone, we get a left-order :

$$g <_P h \Leftrightarrow g^{-1}h \in P.$$

Proposition

- $P = P_{<_P}$
- $g < h \Leftrightarrow g <_{P_<} h$

Definition of $\mathrm{LO}(\Gamma)$

Since left-orders and positive cones are interchangeable,

Definition

The **space of left-orders** of Γ is defined as

$$\mathrm{LO}(\Gamma) := \left\{ P \in 2^{\Gamma} : P \text{ is a positive cone} \right\}$$

LO(Γ) is Compact Polish

Axioms for positive cones are universal $\rightsquigarrow$ $\mathrm{LO}(\Gamma)$ is a closed subspace of 2^{Γ}, hence compact Polish.

$$\forall g, h, (g \in P \wedge h \in P) \rightarrow gh \in P \rightsquigarrow \bigcap_{g,h} U_g^c \cup U_h^c \cup U_{gh}$$

Definition of Archimedean Orders

Definition

We say that an order is **Archimedean** if there is for any $g, h \in P_<$,

$$\exists n, h < g^n$$

We extend this definition to positive cones. We also have a Polish space of Archimedean order, $\mathrm{Ar}(\Gamma)$.

Examples

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are Archimedean-orderable.
- $\Gamma = \langle x, y \mid yxy^{-1} = x^{-1} \rangle$ admits no Archimedean order.

Non-example

WLOG, $x, y \in P$. Suppose there is n with $y < x^n$.

$$\begin{aligned} & y < x^n \\ \Rightarrow & y < y x^{-n} y^{-1} \\ \Rightarrow & \, \mathrm{id} < x^{-n} y^{-1} \\ \Rightarrow & x^n < y^{-1} \end{aligned}$$

But then both y, y^{-1} are positive.

Definition of Conjugacy Action

Definition

The **conjugacy action** $\Gamma \curvearrowright \mathrm{LO}(\Gamma)$ is defined by

$$g \cdot P := g^{-1}Pg = \left\{g^{-1}hg : h \in P\right\}.$$

In particular,

$$g <_{k \cdot P} h \Leftrightarrow gk <_P hk.$$

Definition of Isomorphism Action

The conjugacy action is the restriction of the following action :

Definition

The **isomorphism action** $\mathrm{Aut}(\Gamma) \curvearrowright \mathrm{LO}(\Gamma)$ is defined by

$$\phi \cdot P := \phi(P) = \{\phi(h) : h \in P\} .$$

Both these actions restrict to actions

$$\Gamma \curvearrowright \mathrm{Ar}(\Gamma)$$
$$\mathrm{Aut}(\Gamma) \curvearrowright \mathrm{Ar}(\Gamma)$$

Motivation

We are interested in the complexity of $\mathrm{GL}(\mathbb{Z}^2) \curvearrowright \mathrm{Ar}(\mathbb{Z}^2)$, motivated by

Theorem, Calderoni- Marker- Motto Ros- Shani

$\mathrm{GL}(\mathbb{Q}^2) \curvearrowright \mathrm{Ar}(\mathbb{Q}^2)$ is not smooth.

Still not known whether it is hyperfinite.

What does $\mathrm{Ar}(\mathbb{Z}^2)$ look like ?

Our first goal is to determine what space $\mathrm{Ar}(\mathbb{Z}^2)$ is.

Theorem, folklore

If $P \in \mathrm{LO}(\mathbb{Z}^2)$, there is a line Δ such that $\mathbb{R}^2 - \Delta$ has one component with only positive elements and one component with only negative elements.

If $P \in \mathrm{Ar}(\mathbb{Z}^2)$, $\Delta \cap \mathbb{Z}^2 = \emptyset$.

The big picture

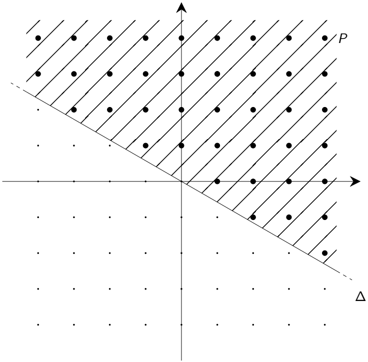

Figure – A positive cone in $\mathbb{Z}^2$. Bigger dots represent elements of P and the shaded region is the half-plane containing only positive elements.

Sketch of Proof

- Observe that one can extend P to $\mathbb{Q}^2$.
- Consider the following partition :
  - $R_1 = \{x \in \mathbb{R}^2 : \exists \epsilon,\ B(x,\epsilon) \cap \mathbb{Q}^2 \subset P\}$
  - $R_2 = \{x \in \mathbb{R}^2 : \exists \epsilon,\ B(x,\epsilon) \cap \mathbb{Q}^2 \subset -P\}$
  - $\Delta\ = \{x \in \mathbb{R}^2 : \forall \epsilon, B(x,\epsilon) \cap \mathbb{Q}^2 \not\subset P, -P\}$
- Prove that R_i are non-empty and open. Prove that Δ is a linear subspace.

Consequences

There is a 2-1 map from $\mathrm{Ar}(\mathbb{Z}^2)$ to line Δ which do not intersect $\mathbb{Z}^2$. This is equivalent to having $\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \in \Delta$, where α is irrational.

We can act as if the map is 1-1, since in each preimage we can pick canonically the cone with $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in P$.

Definition

Definition

The **action by Möbius transformations** is the action GL($\mathbb{Z}^2$) $\curvearrowright$ Irr defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \alpha := \frac{a\alpha + b}{c\alpha + d}$$

Action on lines

If $\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \in \Delta$,

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \Delta \\ \Rightarrow &\begin{pmatrix} a\alpha + b \\ c\alpha + d \end{pmatrix} \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \Delta \\ \Rightarrow &\begin{pmatrix} \frac{a\alpha+b}{c\alpha+d} \\ 1 \end{pmatrix} \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \Delta \end{aligned}$$

Recap

- The action $\mathrm{GL}(\mathbb{Z}^2) \curvearrowright \mathrm{Ar}(\mathbb{Z}^2)$ is bireducible to the action $\mathrm{GL}(\mathbb{Z}^2)$ on lines with irrational slope.
- The action $\mathrm{GL}(\mathbb{Z}^2)$ on lines with irrational slope is bireducible with the action by Möbius transformations.

Continuous fractions

There is an homeomorphism $\mathrm{Irr} \cong \mathbb{N}^{\mathbb{N}}$ defined by

$$[a_0, a_1, ...] := a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + ...}}$$

Goal : Hope that Möbius transformations correspond to a well-studied CBER.

Generators for $\mathrm{GL}(\mathbb{Z}^2)$

We know that $\mathrm{GL}(\mathbb{Z}^2)$ is generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These matrices act nicely through Möbius transformations.

First two matrices

We have that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot [a_0, a_1, ...] = \begin{cases} [a_1, a_2, a_3, ...] \text{ if } a_0 = 0 \\ [\ 0, a_0, a_1, ...] \text{ if } a_0 \neq 0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot [a_0, a_1, ...] = [a_0 + 1, a_1, ...]$$

Chaining these two matrices, we can get tail equivalence relation

$$[a_0, ..., a_n, c_0, c_1, ...] \sim [b_0, ..., b_m, c_0, c_1, ...]$$

Last Matrix

What about $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$?

Theorem, noted in Jackson-Kechris-Louveau, proof found in Hardy-Wright

The orbit equivalence of Möbius transformations on $\mathbb{N}^{\mathbb{N}}$ is exactly tail equivalence relation.

Theorem

Theorem

The isomorphism relation $\mathrm{GL}(\mathbb{Z}^2) \curvearrowright \mathrm{Ar}(\mathbb{Z}^2)$ is hyperfinite, but not smooth.

Question

If $R = \mathbb{Z}[\frac{1}{n}]$ is an intermediate ring, how complicated is $\mathrm{GL}(R^2) \curvearrowright \mathrm{Ar}(R^2)$?

Thank you !